BRINGING THE THE YARROW AND COIN METHODS INTO AGREEMENT

STUART M. ANDERSON

Abstract. Here, I calculate the probabilities of each configuration of lines for the traditional yarrow stalk method and for the simple coin method. I show that these are different and propose two alternative coin methods which reproduce the yarrow stalk method probabilities.

I should state up front that my interest in the I Ching is purely mathematical. I do not use it myself for the casting of oracles, and I have no "feel" for the meaning of the hexagrams. Since that's true, I hope you will forgive me if I do not address questions about which I have nothing of value to say! I will try to be as clear as possible when discussing the mathematical aspects of the I Ching. The mathematics needed consists of three things: basic probability, arithmetic, and a wee bit of number theory. There should be nothing very difficult here.

Let's start with the yarrow stalk method. As a reference, I am following the procedure described in the website http://www.onlineclarity.co.uk/learn/ways-to-consult-the-i-ching/yarrow-method/

Since the method described there is rather complicated (11 steps repeated 3 times) you might expect that a mathematical description of it would also be complicated. Not so! If we first make three crucial observations, the method and its analysis simplify drastically. The first and most important thing to understand is the process of grasping stalks 4 at a time until 4 or fewer remain. This is the same as looking for the remainder when dividing by 4. If the number of stalks in your left hand is N, then if N is divisible by 4, you will end up with 4 stalks between your fingers. If there is a remainder when you divide N by 4, then that remainder will be the number of stalks between your fingers. Mathematically, computing the remainder upon division by 4 is called "reducing modulo 4," and is written "$N \bmod 4$."

Fact 1. *The outcome of each stage of the yarrow stalk method is determined entirely by the reducing the number of stalks in the left hand modulo 4.*

Next, because of Fact 1, if you were to start with 4 more stalks (i.e., you have $N + 4$ in your left hand) you would end up with exactly the same number of stalks between your fingers at the end. That's because if there were 4 extra stalks to start with, you would automatically pull out an extra batch of 4, which would compensate for the change. Of course, you could also add *another* 4, or subtract 4, and so on. None of it makes any diffrerence. Bumping N up or down by 4 (even repeatedly) has *no effect*. Mathematically, this says that $(N + 4) \bmod 4 = N \bmod 4$.

Fact 2. *Numbers of stalks which differ by 4 produce the same final result.*

Date: January 4, 2017,

Third, there is a rule for combining these results. Suppose you have two numbers N and M. If you compute $N \mod 4$ and $M \mod 4$ and add them up, you always get either $(N+M) \mod 4$ or else 4 more than that. Since this is much less clear than Facts 1 and 2, some further explanation is in order: Suppose we write a line of "I"s to represent the stalks. Let's do $N = 17$ and $M = 13$ (these numbers were chosen at random for illustration), graphically dividing them into sets of 4 to see the remainders.

```
N:   (IIII)(IIII)(IIII)(IIII)                    [I] remainder is 1
M:   (IIII)(IIII)(IIII)                          [I] remainder is 1
N+M: (IIII)(IIII)(IIII)(IIII)(IIII)(IIII)(IIII)  [II] remainder is 2
```

N and M each had a remainder of 1, and $N + M$ had a remainder of 2 *because* all we did was to rearrange the sets of 4 into a line and put the remainders from N and M together. That's why the two remainders add up to the total remainder. This illustrates $N \mod 4 + M \mod 4 = (N + M) \mod 4$.

Sometimes the remainders add up to more than 4. In that case, you can pull out another set of 4 stalks. For example, if you took $N = 19$ and $M = 11$, you would get

```
N:  (IIII)(IIII)(IIII)(IIII)                        [III]   remainder 3
M:  (IIII)(IIII)                                    [III]   remainder 3
N+M:(IIII)(IIII)(IIII)(IIII)(IIII)(IIII)   [(IIII)  II]   remainder 2
```

N and M now each have a remainder of 3, and so there are now 6 stalks in the last group, so you can take another 4, leaving 2. Now the remainders add up as $3 + 3 = 2 + 4$. This is what I meant above in Fact 3 when I said "...or 4 more than that." Note that the remainders can never be bigger than 3, so they never add up to more than 6, so you can never pull more than one extra set of 4 out. That's why the only two possibilities are "$(N + M) \mod 4$ or else 4 more than that."

Fact 3. *The remainders of the N stalks initially in the left hand and the M stalks set aside will always sum up to* $(N + M) \mod 4$ *or else 4 more than that.*

I hope I haven't dwelt too long on these preliminaries; since they are crucial to what follows, and since I know from experience that there is a *very* wide range in people's mathematical comfort levels, I have tried to err on the side of overexplaining. Now on to the more interesting stuff: the application of Facts 1–3 to the yarrow stalk method.

First, you always start with 50 stalks, setting 1 aside to leave 49. Then you divide the 49 stalks into two bunches and set one aside, putting 1 stalk between your fingers from the bunch you set aside. That means that the two bunches you intend to count out by 4's contain a total of 48 stalks. Since $48 \mod 4 = 0$, by Fact 3, the remainders from the two bunches add up to 0 or 4. Right away, this simplifies the yarrow method, since it means that once you have the remainder of the first bunch, *you don't need to do the second bunch*, since its remainder is completely determined by Fact 3.

Here is how that works, in tabular form:

First bunch remainder:	0	1	2	3
Second bunch remainder:	0	3	2	1
Add up to:	0	4	4	4
Total number in fingers:	9	5	5	5

Since the remainders are always between 0 and 3, this is the only possible way that they can add up to 0 or 4. (Note: remember that a remainder of 0 means 4 stalks between your fingers.) Therefore, if the first bunch gives a remainder of 0, so will the second bunch, and you are guaranteed to be holding 9 stalks at the end of the process. If the first bunch remainder is anything else, you will be holding 5 stalks. Since 9 stalks has a value of 2 and 5 stalks has a value of 3, the first step of casting the oracle reduces to this formula:

Formula 1. *If the first bunch of stalks is divisible by 4, then the value is 2; otherwise it is 3.*

From here it gets easier. The next step of the oracle casting process repeats the steps of the first step, but uses the stalks set aside in groups of 4 during the first step. Since in the first step, you started with 49 stalks and retained either 5 or 9 in your fingers, the second step starts with either 44 or 40. Again you divide them into two bunches and transfer 1 stalk from the second bunch to your fingers. This means that the stalks in the two bunches total either 43 or 39. Now since $43 = 39+4$, Fact 2 says that there is no difference between these two cases. Changing from 39 to 43 stalks has *no effect*. That means that the outcome of the first step of the oracle casting does not influence the outcome of the second step. They are *independent*. Therefore, we don't have to break the second step down into cases depending on the outcome of the first step, but instead we can treat it exactly the same way as the first step:

Since $39 \mod 4 = 3$, the remainders from the first and second bunches must add up to either 3 or 7 by Fact 3. But there is no way for the two remainders to add up to 7, since each must be less than 4, so the only possibility is that they add up to 3. Once again, the remainder from the second bunch is completely determined once you have counted out the first bunch. The following table shows the only possible ways the remainders can turn out:

First bunch remainder:	0	1	2	3
Second bunch remainder:	3	2	1	0
Add up to:	3	3	3	3
Total number in fingers:	8	4	4	8.

Therefore, if the first bunch remainder is 0 or 3, you will end up holding 8 stalks, and if it is 1 or 2, you will have 4 stalks. Since 8 stalks has a value of 2 and 4 stalks has a value of 3, the second step of casting the oracle reduces to the formula:

Formula 2. *If the remainder of the first bunch is 0 or 3, the value is 2; otherwise it is 3.*

Now as to the third step, it is very similar to the second. Since the second step started with 39 or 43 stalks, and you end up with 4 or 8 in your fingers, the third step could start with any of the three possibilities: $31 = 39 - 8$, or $35 = 39 - 4 = 43 - 8$, or $39 = 43 - 4$. Since all these differ from one another by multiples of 4, the same reasoning we applied in the second step shows that all these numbers act the same, and there is no need to break the third step down into cases either.

Since $31 \mod 4 = 3$, which is the same result we got for the second step, the table for the third step is identical to the one for the second step, and the same formula follows.

Now all this means that the stalk method can be reduced to the following process (where choosing a random number is the same as grabbing a random number of stalks):

Method 1 (Mathematical Yarrow Stalk Method)**.** *Four steps:*

Step 1: Choose a random number and divide by 4. If the remainder is 0, then the first value is 2; otherwise it is 3.

Step 2: Choose a random number and divide by 4. If the remainder is 0 or 3, then the second value is 2; otherwise it is 3.

Step 3: Repeat Step 2.

Step 4: Add the three values obtained above. The result will be 6, 7, 8, or 9; 6 and 8 are weak, 7 and 9 are strong; 6 and 9 are changing, 7 and 8 are unchanging.

Now a word about the probabilities: it is *reasonable* but not *exact* to assume that the probability of each remainder is $\frac{1}{4}$. It is true that exactly $\frac{1}{4}$ of *all* numbers have a remainder of 0, $\frac{1}{4}$ have a remainder of 1, etc., *but* with a finite list of numbers this is no longer true. For 31 stalks, for example, the numbers with each remainder are

```
0:      4,8,12,16,20,24,28,      (7 of them)
1:      1,5,9,13,17,21,25,28,    (8 of them)
2:      2,6,10,14,18,22,26,30,   (8 of them)
3:      3,7,11,15,19,23,27,31.   (8 of them).
```

If each number is equally likely to be chosen, then the probabilities of 0,1,2,and 3 are respectively $\frac{7}{31}$, $\frac{8}{31}$, $\frac{8}{31}$, and $\frac{8}{31}$. These are close to $\frac{1}{4}$, but not exact.

However, there is a human tendency that puts the probabilities back very close to $\frac{1}{4}$. If you were asked to divide a bunch of stalks into two sets by grabbing a bunch of them, would you ever grab all of them? How about none of them? How about just 1 or all but 1? It somehow doesn't seem "fair" or "random" to divide them like that. You'd probably divide them into nearly equal shares by grabbing somewhere between $\frac{1}{3}$ and $\frac{2}{3}$ of them. If you never go near the extremes it doesn't matter that you're working with a finite list of numbers. The probability will therefore be almost exactly $\frac{1}{4}$ for each remainder.

Suppose now (as seems reasonable) that the probabilities are $\frac{1}{4}$. Then we can calculate the probability of each type of line. From the first step, we get a value of 2 a quarter of the time and 3 the rest of the time. From the second and third steps, we get a value of 2 half the time, and 3 the other half of the time. Then we can calculate the probabilities:

```
p(6) =                       p1(2)*p2(2)*p3(2)
                          =  (1/4)*(1/2)*(1/2)
                          =  1/16,

p(7) =  p1(2)*p2(2)*p3(3) + p1(2)*p2(3)*p3(2) + p1(3)*p2(2)*p3(2)
     =  (1/4)*(1/2)*(1/2) + (1/4)*(1/2)*(1/2) + (3/4)*(1/2)*(1/2)
     =  5/16,

p(8) =  p1(2)*p2(3)*p3(3) + p1(3)*p2(3)*p3(2) + p1(3)*p2(2)*p3(3)
     =  (1/4)*(1/2)*(1/2) + (3/4)*(1/2)*(1/2) + (3/4)*(1/2)*(1/2)
     =  7/16,
```

```
p(9) =                     p1(3)*p2(3)*p3(3)
                        =  (3/4)*(1/2)*(1/2)
                        =  3/16.
```

In the above list, $p(X)$ stands for the probability of getting a total value of X, and $p_1(X)$, $p_2(X)$, and $p_3(X)$ stand for the probabilities of getting a value of X in the first, second, and third steps of the casting process. Recall that when you want several things to be true in combination (such as getting values of 2,2,2 on the three steps), you *multiply* their individual probabilities, and where there are several alternatives (such as getting 2,2,3 OR 2,3,2 OR 3,2,2), you *add* their indvidual probabilities. That is how the above calculations were done. We can fit these values into a table as follows:

	Strong	Weak
Changing	9	6
Unchanging	7	8

This gives a table of probabilities for strong vs. weak and changing vs. unchanging:

	Strong	Weak	Total
Changing	3/16	1/16	1/4
Unchanging	5/16	7/16	3/4
Total	1/2	1/2	

The rightmost column shows the total probabilities to be changing or unchanging. $\frac{1}{4}$ of the lines will be changing, $\frac{3}{4}$ of them unchanging. The bottom row shows the total probabilities to be strong or weak. $\frac{1}{2}$ of the lines will be strong, $\frac{1}{2}$ of them weak.

Now for comparison, the coin method (much, much simpler than the stalk method!):

Method 2 (Original Coin Method)**.** *Two steps:*

Step 1: Toss 3 coins. Heads are worth 2, tails worth 3.

Step 2: Add the three values obtained above. The result will be 6, 7, 8, or 9; 6 and 8 are weak, 7 and 9 are strong; 6 and 9 are changing, 7 and 8 are unchanging.

The coin method gives probabilities as follows:

```
     HHH       = 6
HHT, HTH, THH = 7
HTT, THT, TTH = 8
     TTT       = 9
```

Since there are 8 possibilities, the probabilities are as follows:

```
p(6) = 1/8 (one way to get 6),
p(7) = 3/8 (three ways to get 7),
p(8) = 3/8 (three ways to get 8),
p(9) = 1/8 (one way to get 9).
```

In the same table format as for the stalk method, the probabilities are

	Strong	Weak	Total
Changing	1/8	1/8	1/4
Unchanging	3/8	3/8	3/4
Total	1/2	1/2	

Notice that the Total column and Total row are the same as for the stalk method, but the body of the table is different. This means that $\frac{1}{4}$ of the lines will be changing and the rest unchanging, and $\frac{1}{2}$ will be strong and the other half weak, just as with the stalk method; however, *the individual distributions of properties are different.*

There is a statistical test that one can apply to probabilities when there are two properties under study. When the product of each row total with each column total agrees with each entry in the body, then the two properties are statistically *independent*. The coin method satisfies this test, since for example the Changing total times the Strong total is $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$, which agrees with the Strong Changing entry in the body. It works for all the other entries, too. But it doesn't work for the yarrow stalk method. There is a statistical correlation in that changing lines are usually strong, whereas unchanging lines are usually weak in the yarrow stalk method. Strong and Changing are *independent in the coin method, not independent in the yarrow stalk method.*

By the way, since the probabilities are independent in the coin method, you can revise the Original Coin Method to reflect that independence:

Method 3 (Equivalent Coin Method)**.** *Two steps:*
Step 1: Toss 2 coins. If both are heads, the line is changing.
Step 2: Toss 1 coin. If it is heads, the line is strong.

The outcome is identical to that of the usual coin method, but notice that the first coin toss determines whether the line is changing and the second one determines whether it is strong. Since these are two separate coin tosses, the outcome of the first toss can in no way affect the second, so the two properties are clearly independent. This is no "improvement" over the usual coin method; it is just here to illustrate the fact of independence.

Now that we have the probability table worked out for both methods, we can redesign the coin method to bring it into agreement with the stalk method. Going by the usual route of getting values and adding them up, we could do as follows:

Method 4 (Alternative Coin Method 1)**.** *Three steps:*
Step 1: Toss two coins. If both are heads, write 2; otherwise 3.
Step 2: Toss two coins. Each head is worth 2, each tail 3.
Step 3: Add up the results, to get a number between 6 and 9.

The reason this works is that the probability of getting a remainder of 0 in the stalk method is $\frac{1}{4}$, and the probability of getting 2 heads is also $\frac{1}{4}$. The probability of getting a remainder of 0 or 3 is $\frac{1}{2}$ (2 chances out of 4 remainder possibilities), which is the same as the probability of getting a head when flipping *one* coin. Steps 2 and 3 of the mathematical yarrow stalk method have been combined in step 2 of this alternate coin method, since they are identical. Here we throw two coins and consider them separately instead of throwing one coin and then throwing it again.

Or skipping the calculation of values, and getting straight to the strong/weak and changing/unchanging properties:

Method 5 (Alternative Coin Method 2)**.** *Two steps:*
Step 1: Toss 1 coin. Heads means strong, tails means weak. Leave the coin where you tossed it.
Step 2: Toss 3 more coins next to the first one. If you see exactly 3 heads among all 4 coins, then the line is changing; otherwise unchanging.

While the Alternative Coin Method 1 was a step by step translation of the stalk method into coin tosses, Alternative Coin Method 2 was designed from scratch to give the same probabilities as 1. Here is how the cases break down (below). As you can see, there are 7 weak unchanging cases, 5 strong unchanging cases, 3 strong changing cases, and 1 weak changing case, out of a total of 16 cases. This gives the probabilities exactly as in the table above for the yarrow stalk method.

H	HHH	strong	unchanging	T	HHH	weak	changing
H	HHT	strong	changing	T	HHT	weak	unchanging
H	HTH	strong	changing	T	HTH	weak	unchanging
H	HTT	strong	unchanging	T	HTT	weak	unchanging
H	THH	strong	changing	T	THH	weak	unchanging
H	THT	strong	unchanging	T	THT	weak	unchanging
H	TTH	strong	unchanging	T	TTH	weak	unchanging
H	TTT	strong	unchanging	T	TTT	weak	unchanging

By the way, it was very tricky to figure out what the two rules of alternate method 2 should be, because the properties are not statistically independent. There was no very simple pattern to the cases, and I had to stare at it for a long time before I found a verbal way to phrase the rules that would both sound simple and yet get the probabilities right.

Of the two alternate coin methods, I personally like 2 better, since it directly gives you the line without adding up any numbers. All you have to do is count the heads. Some people may prefer alternate coin method 1 because it adheres more closely to the yarrow stalk method.

One final note on the alternate coin methods. Both involve 4 coin tosses, rather than 3 as in the usual coin method. This is absolutely necessary, since the yarrow stalk method probabilities are all fractions with 16 in their denominator. 4 coins must be tossed to generate 16 cases. In fact, this constitutes an airtight mathematical proof that the standard coin method cannot possibly give the same probabilities as the yarrow stalk method. *Any* method that involves only 3 coins cannot work, since the probabilities will have denominators of 8, and therefore cannot be equal to the probabilities of the yarrow stalk method.